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# Not Every Finite Lattice is Embeddable in the Recursively Enumerable Degrees*

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A certain lattice with eight elements is shown to be not embeddable as a lattice in the recursively enumerable degrees. This refutes the well-known Embedding Conjecture which asserted that every finite lattice could be so embedded.

Since the papers of Kleene, Post, and Spector [2, 10, 20], a great deal of important research has been devoted to the degrees $\mathscr{D}$ of unsolvability, particularly their elementary theory, Th($\mathscr{D}$) and their algebraic structure as an upper semi-lattice. Lachlan [4] proved undecidability of Th($\mathscr{D}$) by proving that all finite lattices could be embedded as initial segments of degrees. Simpson [18] later obtained a more exact classification by showing Th($\mathscr{D}$) to be recursively isomorphic to the truth set of second order arithmetic. Building on a technique of Lerman [7], Lachlan and Lebeuf [6] showed that every countable upper semi-lattice could be embedded in $\mathscr{D}$ as an initial segment, and Nerode and Shore [9] used this to obtain a simple proof of Simpson's result.

Among classes of degrees, those containing recursively enumerable sets (the r.e. degrees) have received particular attention since the time of Post [10] partly because of the widespread occurrence of r.e. sets in algebra and number theory as explained in [19], and their application in famous undecidability results. Both the decision problem for the elementary theory and embedding questions are much more difficult for the r.e. degrees $\mathscr{R}$ than for $\mathscr{D}$. For example, Friedberg [1] and Muchnik [8] invented the priority method just to prove that there are more than two r.e. degrees, and the proof that $\mathscr{R}$ is not a lattice was only given in [3]. Sacks [14] showed that the r.e. degrees are dense, probably the most pleasing property of $\mathscr{R}$ to have been proved. Many other elementary properties of $\mathscr{R}$ have been proved, some by the exercise of great ingenuity, but the overall picture has remained obscure. A survey of the literature can be found in [19].

Here we are chiefly concerned with what finite lattices can be embedded in $\mathscr{R}$ as lattices, i.e., we are insisting that both joins and meets be preserved by

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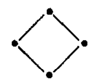
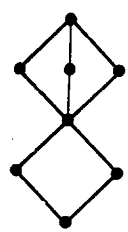
the embedding. The first result along this line, due independently to Lachlan [3] and Yates [22], showed that the lattice



can be embedded in $\mathscr{R}$ as a lattice with least element mapped to $\mathbf{0}$. Building on this Thomason [21] and Lachlan [5], and independently, Lerman (unpublished) showed that all countable distributive lattices can be embedded in $\mathscr{R}$ as lattices. Using a much more complex technique of enumeration Lachlan [5] showed how the two 5-element nondistributive lattices can be embedded in $\mathscr{R}$ as lattices. This led many to attempt to prove the Embedding Conjecture [11, 16, 19] which asserts that every finite lattice can be embedded in $\mathscr{R}$ as a lattice. A solution of this conjecture is necessary to decide even which *existential* sentences are true in $\mathscr{R}$.

In this paper we establish that the lattice $\mathscr{L}$ (also called $S_8$) with diagram



is not embeddable in $\mathscr{R}$ as a lattice. Lerman was the first to suggest that the Embedding Conjecture might be false and in particular that $S_8$ might not be embeddable. Although great difficulties remain, we hope this will at least point the way to a solution of the whole embedding problem for $\mathscr{R}$, and will thus shed some light on the decision problem for its elementary theory.

*Preliminaries.* Below we shall deal with sets, functions, and functionals which are enumerated in $\omega$ stages. We think of all these enumerations as occurring simultaneously. For brevity we deliberately employ an ambiguous notation. If $A$ denotes an r.e. set then it also denotes the particular enumeration of the r.e. set, and within the context of a stage denotes the current approximation to $A$. If the stage is not clear then the approximation to $A$ at stage $s$, i.e., the finite set enumerated in $A$ before stage $s$, is denoted $A[s]$. The sequence $\langle A[s]; s < \omega\rangle$ is increasing and strongly r.e. Finally, $A$ also denotes the characteristic function of $A$. If $\Psi$ denotes a p.r. functional $\Psi$ also denotes the enumeration of $\Psi$: $\langle \Psi[s]: s < \omega\rangle$ a strongly r.e. increasing sequence of finite functionals. In the context of stage $s$, $\Psi$ refers to $\Psi[s]$ the current approximation.

For functions our terminology is unorthodox. Functions have natural-number arguments but are not necessarily total. If $\varphi$ is a function $\varphi$ also denotes the particular enumeration of it with which we are concerned. But in this case $\langle\varphi[s]: s < \omega\rangle$ is an r.e. sequence of p.r. functions such that $x \in \operatorname{dom} \varphi[y]$ is a recursive binary relation and for all $x, y$

$$\varphi(x) = y \leftrightarrow \exists t \forall s(s \geqslant t \rightarrow \varphi(x)[s] = y).$$

We are not requiring that $\varphi[s]$ be finite nor that $\varphi[s+1]$ extend $\varphi[s]$. Clearly, the total functions enumerable in this sense are those of degree $\leqslant \mathbf{0}'$. If $A$ is an r.e. set and $\varphi$ is a function, then $\varphi$ *respects* $A$ means that for all $x, s$

$$\begin{aligned}&\varphi(x)[s] \text{ defined } \& \ (\varphi(x)[s+1] \text{ undefined or } \phi(x)[s+1] \neq \phi(x)[s]) \\ &\quad .\!\rightarrow\!. \ \exists y(y < \varphi(x)[s] \ \& \ y \in A[s+1] - A[s]).\end{aligned}$$

For any functional $\Psi$ and set $A$ we define the *use function* $\psi$ *of* $\Psi$ *with respect to* $A$ by:

$$\begin{aligned}\psi(x)[s] = \max(&\{\mu y(\Psi(A \restriction y)(x)[s] \text{ is defined}) \cup \{x\} \\ &\cup \{\psi(x)[t]: t < s \ \& \ \psi(x)[t] \text{ is defined}\}).\end{aligned}$$

Note that $\psi$ respects $A$ and that $\psi$ is total if $\Psi(A)$ is total.

For any set $A$ its degree will be denoted $\mathbf{a}$, i.e., by the corresponding lower case letter.

We now move directly to the proof of

THEOREM. *$\mathscr{L}$ cannot be embedded as a lattice in the upper semilattice of* r.e. *degrees.*

*It suffices to prove something apparently weaker, namely, that*

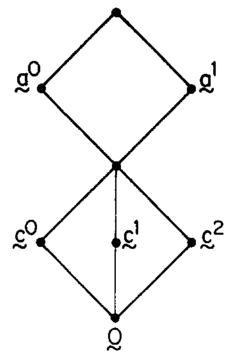
$$\textit{$\mathscr{L}$ cannot be embedded with its least element $0_{\mathscr{L}}$ being mapped to $\mathbf{0}$.} \qquad (1)$$

From the latter by relativization for any r.e. degree $\mathbf{b}$ there is no embedding of $\mathscr{L}$ as a lattice into the $\mathbf{b}$-r.e. degrees with $0_{\mathscr{L}}$ being mapped to $\mathbf{b}$. From [2, p. 568] we have:

LEMMA 1. *Let $\mathbf{a}_0$, $\mathbf{a}_1$ be* r.e. *degrees and $\mathbf{b}$ a degree not necessarily* r.e. *such that $\mathbf{b} \leqslant \mathbf{a}_0$, $\mathbf{a}_1$; then there exists an* r.e. *degree $\mathbf{c} \leqslant \mathbf{a}_0$, $\mathbf{a}_1$ such that $\mathbf{b} \leqslant \mathbf{c}$.*

An immediate consequence is that an embedding of $\mathscr{L}$ as a lattice into the r.e. degrees with $0_{\mathscr{L}}$ going to $\mathbf{b}$ would simultaneously be an embedding into the $\mathbf{b}$-r.e. degrees. Thus below we need only prove (1).

For proof by contradiction, suppose given r.e. sets $A^0$, $A^1$ $C^0$, $C^1$, $C^2$ which witness the embedding of the lattice $\mathscr{L}$ in the upper semilattice of r.e. degrees in the manner of the diagram:




It is convenient to suppose that $C^i$ contains only even numbers congruent to $i$ (mod 3). Let $C = \bigcup\{C^i: i < 3\}$ then $C$ contains only even numbers and we may assume that $A_0$ and $A_1$ agree with $C$ on the even numbers. For $\{i, j, k\} = \{0, 1, 2\}$ let $\tau^i$ be the enumeration of a function witnessing $\mathbf{c}^i \leqslant \mathbf{c}^j \cup \mathbf{c}^k$ in the sense that

(i) $\tau^i[s]$ is total and converges as $s \to \infty$;

(ii) $\tau^i$ respects $C^j \cup C^k$;

(iii) for all $x$ and $s$, $\tau^i(x)[s]$ is increasing in $x$ and nondecreasing in $s$, and $x < \tau^i(x)[s]$;

(iv) for all $x$, $s$

$$x \in C^i[s+1] - C^i[s] \to \exists y(y < \tau^i(x)[s] \ .\&.\ y \in C^j[s+1] - C^j[s] \vee y \in C^k[s+1] - C^k[s]).$$

Let $\tau(x) = \max\{\tau^i(x): i < 3\}$, then $\tau$ respects $C$.

To verify the existence of the enumerations $\tau^i[s]$ and $C^i[s]$, fix recursive functionals $\Phi^i$ such that $C^i = \Phi^i(C^j \cup C^k)$, and let $\varphi_i$ be the use function for $\Phi_i$. Let $\hat{C}^i[s]$, $s \in \omega$, be any recursive enumeration of $C^i$, and define recursive functions

$$l^i(s) = (\mu x)(\forall y < x)[\hat{C}^i[s] = \Phi^i(\hat{C}^j \cup \hat{C}^k)[s]].$$

Choose stages $t_0 < t_1 < \cdots$ such that

$$(\forall i < 3)[\hat{C}^i[t_{n+1}] - \hat{C}^i[t_n] \neq \varnothing \qquad \text{and} \qquad n \leqslant l^i(t_n)].$$

Let $z_n{}^i = \mu x[x \in \hat{C}^i[t_{n+1}] - \hat{C}^i[t_n]]$. Define $C^i[s] = \hat{C}^i[t_s]$, $g(i) = (i+1)$ mod 3, and

$$\begin{aligned} \sigma^i(x)[s] &= \varphi_i(x)[t_s] && \text{if} \quad x \leqslant s \\ &= 1 + z_s^{g(i)} && \text{if} \quad x > s. \end{aligned}$$

Define $\tau^i(x)[s] = \max\{1 + x, \sigma^i(x)[s], \tau^i(x)[s-1]\}$. Clearly, these enumerations satisfy (i), (ii), and (iii) above. For (iv), assume $x \in C^i[s+1] - C^i[s]$. If $x > s$, then

$$\tau^i(x)[s] > z_s^{g(i)} \in (C^j \cup C^k)[s+1] - (C^j \cup C^k)[s].$$

If $x < s$, then $\tau^i(x)[s] \geqslant \varphi^i(x)[s]$. But $x \in \hat{C}^i[t_{s+1}] - \hat{C}^i[t_s]$, $l^i(t_s) \geqslant s \geqslant x$, and $l^i(t_{s+1}) \geqslant s + 1 \geqslant x$ imply there is some $y < \varphi^i(x)[s]$, such that

$$y \in (\hat{C}^j \cup \hat{C}^k)[t_{s+1}] - (\hat{C}^j \cup \hat{C}^k)[t_s].$$

Let $\theta^0$, $\theta^1$, $\psi$ be enumerations of functions. With respect to this triple of enumerations we say *$n$ is active at stage $s$* if $\theta^0(n)$, $\theta^1(n)$, and $\psi(n)$ are all defined and

$$\tau\psi(n) < \theta^0(n) = \theta^1(n).$$

Call *$n$ permanently active* if, for some $s$, $n$ is active all stages $\geqslant s$.

LEMMA 2. *Given a recursive enumeration $\psi$ of a use function which respects $C$ we can effectively enumerate, uniformly in $C$, total functions $\theta^0$, $\theta^1$, respecting $A^0$, $A^1$, respectively, such that if $\psi$ is total then there are infinitely many $n$ which are permanently active. Further, for all $n$ and $s$*

$$(n \text{ active at stage } s \text{ but not at stage } s+1) \to$$
$$\exists x (x < \psi(n)[s] \;\&\; x \in C[s+1] - C[s]).$$

*Proof.* Define $\theta^0(x)[0] = \theta^1(x)[0] = x$. If $i < 2$, $s > 0$, $\psi(x)[s-1] = \psi(x)[s]$ is defined, $\theta^i(x)[s-1] > \tau\psi(x)[s-1]$, and $\tau\psi(x)[s] > \tau\psi(x)[s-1]$, then let

$$\theta^i(x)[s] = \tau\psi(x)[s] + 1.$$

If $s > 0$, $\psi(x)[s-1] = \psi(x)[s]$ is defined, $\theta^0(x)[s-1] \leqslant \tau\psi(x)[s]$, and $(A^0[s] - A^0[s-1]) \cap x \neq \varnothing$, let

$$\theta^0(x)[s] = \tau\psi(x)[s] + 1.$$

If $s > 0$, $\psi(x)[s-1] = \psi(x)[s]$ is defined, $\theta^0(x)[s-1] > \tau\psi(x)[s]$, and $(A^1[s] - A^1[s-1]) \cap x \neq \varnothing$, let

$$\theta^1(x)[s] = \theta^0(x)[s-1].$$

In all other cases if $i < 2$ and $s > 0$ let $\theta^i(x)[s] = \theta^i(x)[s-1]$.

If $\tau\psi(x)[s] > \tau\psi(x)[s-1]$ while $\psi(x)[s] = \psi(x)[s-1]$ then $(C[s] - C[s-1]) \cap \tau\psi(x)[s-1] \neq \varnothing$ since $\tau$ respects $C$. From this it is clear that $\theta^0$, $\theta^1$ respect $A^0$, $A^1$, respectively.

Observe that if at some stage $\theta^i(x) > \tau\psi(x)$ then in fact $\theta^i(x) = \tau\psi(x) + 1$ because $\psi(x)[s] \geqslant x$ and $\tau(y)[s] > y$, so $\theta^i(x) < \tau(\psi(x))$ at $s = 0$. Further, if $\theta^i(x)[s] = \tau\psi(x) + 1$ and $\theta^i(x)[t] = \tau\psi(x)[t] + 1$ with $s < t$, we cannot have $\psi(x)[t] > \psi(x)[s]$ unless some number $< x$ enters $A^i$ at a stage $v, s \leqslant v \leqslant t$. For consider the least $u > s$ with $\psi(x)[u] > \psi(x)$ and let $v$ be greatest such that $s \leqslant v < u$ and $\psi(x)[v]$ is defined, then

$$\tau\psi(x)[u] \geqslant \tau\psi(x)[v] + 1 \geqslant \theta^i(x)[v] = \theta^i(x)[u].$$

Since $\tau\psi(x)[w]$ is nondecreasing in $w$ from the definition of $\theta^i$, $\theta^i(x)[t] > \tau\psi(x)[t]$ implies that some number $<x$ enters $A^i$ at a stage $\geqslant u$ and $<t$. The above observation together with the convergence of $\tau$ guarantees the convergence of $\theta^i$ for $i < 2$.

Suppose $\psi$ is total. For infinitely many $n$ there exists $s$ such that at stage $s$, $\psi(n)$ is defined, $C$ has converged on $\psi(n)$, but $A^0$ has not yet converged on $n$. Otherwise, $A^0$ would be recursive in $C$. Thus using an $A^0$-oracle we can effectively enumerate an infinite sequence $\langle(n(i), s(i)) \colon i < \omega\rangle$ such that for all $i$, $n(i) < n(i + 1)$, $C[s(i)]$ and $C$ agree on $\psi(n(i))$, $A^0[s(i)]$ and $A^0$ agree on $n(i)$, and $\theta^0(n(i))[s(i)] > \tau\psi(n(i))[s(i)]$. Clearly, $\psi(n(i))[s] = \psi(n(i))[s(i)]$ for all $s \geqslant s(i)$ because $\psi$ respects $C$, whence also $\theta^0(n(i))[s] > \tau\psi(n(i))[s]$ for all $s \geqslant s(i)$. There are infinitely many $i$ such that at stage $s(i)$, $A^1$ has not yet converged on $n(i)$. Otherwise, $A^1$ would be recursive in $A^0$. If $x < n(i)$ is enumerated in $A^1$ at stage $t - 1 \geqslant s(i)$ then by definition of $\theta^1$, $\theta^1(n(i))[t] = \theta^0(n(i))[t]$. Further, $\theta^1(n(i))[s] = \theta^0(n(i))[t]$ for all $s \geqslant t$. Thus $n(i)$ is permanently active. This completes the proof of the lemma.

Let $\Psi$ be a recursive enumeration of a functional and $\psi$ be the use function of $\Psi$ with respect to $C$. Then $\psi$ respects $C$. Let $\theta^0$, $\theta^1$ be the enumerations obtained by Lemma 2 from $\psi$. We effectively enumerate a function $\varphi$ as follows:

$$\varphi(x)[0] = 0.$$

If $n$ is active at stage $s$, $\Psi(C)(n)[s]$ is defined, and

$$\exists x(\psi(n) \leqslant x < \tau\psi(n) \;\&\; x \in C[s + 1] - C[s]),$$

then $\varphi(n)[s + 1] = 1 + \Psi(c)[n](s)$. Otherwise, $\varphi(n)[s + 1] = \varphi(n)[s]$.

LEMMA 3. *$\varphi$ is total and is recursive in both $A^0$ and $A^1$ uniformly in $\Psi$, $\psi$.*

*Proof.* Note from Lemma 2 that $\theta^0$, $\theta^1$ are recursive in $A^0$, $A^1$, respectively, uniformly in $\psi$. For $i = 0, 1$, once $\theta^i(n)$ has settled down and all members $<\theta^i(n)$ have entered $C$, $\varphi(n)$ cannot change. This is enough.

Fix a simultaneous recursive enumeration $\langle(W_e^0, W_e^1, W_e^2) \colon e < \omega\rangle$ of all triples of r.e. sets and a recursive partition of $\omega$ into infinite sets $R_e^i$, $(i, e) \in 3 \times \omega$. Members of $R_e^i$ are called *$(i, e)$-numbers*.

With respect to the triple of enumerations $\theta^0$, $\theta^1$, $\psi$ we define enumerations of r.e. sets $G^0$, $G^1$, $G^2$ as follows. Let $\{i, j, k\} = \{0, 1, 2\}$, then enumerate $x$ in $G^i$ at stage $t$ if there exist $e$, $s$, $t$, $n$ such that the following conditions hold:

(1) $x$ is an $(i, e)$-number, $x \in W_e^i[s]$;

(2) $s < t$, $\tau\psi(n)[s] \leqslant x$;

(3) $n$ is active at all stages $\geqslant s$ and $< t$;

(4) $t$ is the least number $> s$ such that $(C[t] - C[s]) \cap \psi(n) \neq \varnothing$,

$$(C^j[t] - C^j[s]) \cap \tau\psi(n)[s] \neq \varnothing$$

and

$$(C^k[t] - C^k[s]) \cap \tau\psi(n)[s] \neq \varnothing.$$

LEMMA 4. *Let $\Psi$ be an enumeration of a functional, $\psi$ be the use function of $\Psi$ with respect to $C$. Let $\theta^0$, $\theta^1$ be generated from $\psi$ by Lemma 2 and then let $\varphi$ and $G^0$, $G^1$, $G^2$ be defined as just described. One of the following three possibilities holds*:

(P1) *$\Psi(C)$ is not total.*

(P2) *$\Psi(C)$ is total and $\Psi(C) \neq \varphi$.*

(P3) *for all $i, j, k$ such that $\{i, j, k\} = \{1, 2, 3\}$ $G^i$ is recursive in both $C^j$ and $C^k$; further, one of $G^0$, $G^1$, $G^2$ is nonrecursive.*

*Proof.* For argument by contradiction, suppose (P1), (P2), (P3) all fail. Let $\{i, j, k\} = \{1, 2, 3\}$. Suppose $G^0$, $G^1$, $G^2$ are all recursive. Then there exists $e$ independent of $i$ such that $W_e^i = \omega - G^i$ and no $(i, e)$-number is ever enumerated in $G^i$. Define $\lambda^i(s)$ to be the greatest $(i, e)$-number in $W_e^i[s]$ and let $\lambda(s) = \min\{\lambda^i(s): i < 3\}$. Then $\lambda(s)$ is defined for all sufficiently large $s$ is nondecreasing and unbounded as $s$ increases. Note that $\lambda(s)$ is a recursive function.

Fix $n$ and $s$ such that $\tau\psi(n)[s] \leqslant \lambda(s)$ and $n$ is active at stage $s$. Then $n$ is active at all stages $\geqslant s$. Otherwise, let $t$ be the first stage $> s$ at which $n$ is not active. Then from Lemma 2

$$\exists x(x < \psi(n)[s] \ \&\ x \in C[t] - C[s]).$$

From this it follows that two of $C^0[t] - C^0[s]$, $C^1[t] - C^1[s]$, and $C^2[t] - C^2[s]$ intersect $\tau\psi(n)[s]$. Hence, for some $i < 3$, $\lambda^i(s)$ is enumerated in $G^i$ making $G^i \cap W_e^i \neq \varnothing$, contradiction. It is also clear that if $\tau\psi(n)[s] \leqslant \lambda(s)$ and $n$ is active at stage $s$ then no number $< \psi(n)[s]$ is enumerated in $C$ after stage $s$. Since $\psi(n) \geqslant n$ whenever $\psi(n)$ is defined and there are infinitely many permanently active $n$ from Lemma 2, we have $C$ recursive. Thus one of $G^0$, $G^1$, $G^2$ is nonrecursive.

Now we turn to the proof that $G^i$ is recursive in both $C^j$ and $C^k$. To this end, observe that there is a 4-ary recursive function $\pi$ with the following property. If $u > s$, $n$ is active at all stages $\geqslant s$ and $\leqslant u$, $y < \tau\psi(n)[s]$, $y \in C[u] - C[s]$, and

$$\neg\exists x(x < \psi(n)[s] \;\&\; x \in C[u] - C[s])$$

then $\pi(s, u, n, y)$ is the least $t > s$ such that

$$\exists x(x < \psi(n)[s] \;\&\; x \in C[t] - C[s]).$$

Such $t$ must exist because otherwise $\varphi(n) \neq \Psi(C)(n)$ which would mean that one of (P1), (P2) holds. For other quadruples $s$, $u$, $n$, $y$ let $\pi(s, u, n, y) = 0$. We can now show that $G^i$ is recursive in $C^j$ as follows. Given $x$ find $v$ such that $C^j[v]$ and $C^j$ agree on $x$. Let

$$v' = \max\{\pi(s, v, n, y)\colon y \in C^j[v] - C^j[s],\ s < \omega,\ n < \omega,\ y < \omega\}.$$

Since $\pi(s, v, n, y) = 0$ for $s \geqslant v$, and $\langle C^j[s]\colon s < \omega\rangle$ and $\langle \mathrm{dom}\ \psi[s]\colon s < \omega\rangle$ are strongly r.e. sequences of finite sets, $v'$ exists and may be effectively computed from $v$. Then $x \in G^i$ if and only if $x \in G^i[v']$ by clause (3) in the definition of $G^i$. This completes the proof that $G^i$ is recursive in $C^j$ and also the proof of Lemma 4.

Since (P3) is impossible from our assumption that $\mathbf{c}^0 \cap \mathbf{c}^1 = \mathbf{c}^1 \cap \mathbf{c}^2 = \mathbf{c}^2 \cap \mathbf{c}^0 = \mathbf{0}$, the argument given above shows that for any p.r. functional $\Psi$ we can effectively enumerate uniformly in $\Psi$ a function $\varphi$ which is recursive in both $A^0$ and $A^1$ uniformly in $\Psi$. Further, $\varphi \neq \Psi(C)$. Let $\langle \Psi_i\colon i < \omega\rangle$ be a standard enumeration of the p.r. functionals and $x \mapsto ((x)_0, (x)_1)$ be a recursive bijection from $\omega$ to $\omega \times \omega$. Define $\delta(x) = \varphi_{(x)_0}(x)_1$, where $\varphi_i$ comes from $\Psi_i$ by our uniform method of enumerating $\varphi$ from $\Psi$. Now $\delta$ is recursive in both $A_0$ and $A_1$. Were $\delta$ recursive in $C$, there would be an r.e. sequence of p.r. functionals $\langle \Phi_i\colon i < \omega\rangle$ such that $\varphi_i = \Phi_i(C)$ for all $i < \omega$. But from the recursion theorem $\Phi_i = \Psi_i$ for some $i$ which contradicts $\varphi_i \neq \Psi_i(C)$. Thus $\delta$ is not recursive in $C$ and we have a degree $\mathbf{d} \leqslant \mathbf{a}^0, \mathbf{a}^1$ with $\mathbf{d} \nless \mathbf{c}$. From Lemma 1 $\mathbf{c} \neq \mathbf{a}^0 \cap \mathbf{a}^1$ in the upper semilattice of r.e. degrees which is the final contradiction we have been seeking.

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